§ **2** *VECTORS FUNCTIONS*

We start with the *general* mathematical concept of a function. There are several ways to formalize the idea of a function. The following is not the most precise possible definition, but it will suffice for this class.

2.1 Definition. Let X and Y be arbitrary sets. A *function* from X to Y is an assignment of a specific member of Y to each member of X.

Like everything else in mathematics, we give functions names in order to talk about them. We often use the letters f, g, and h for functions, but actually any name can be used. More important is the standard use of "functional notation." If f is a function from X to Y and if $x \in X$ then we use the notation $f(x)$ to denote the member of Y which is assigned to x by f. (The element symbol $\in$ denotes membership—$x \in X$ is read "x is in X" and just means that x belongs to the set X.)

2.2 Example. To make this concrete, we review a trivial example. Let f be the function from $\mathbb{R}$ to $\mathbb{R}$ which multiplies each number in $\mathbb{R}$ by itself. Then $f(0) = 0$, $f(2) = 4$, $f(3) = 9$, etc. In general, for any number x, $f(x) = x^2$, and of course this equation is a more convenient way to define this f than the sentence we used about multiplication.

Often, books define a function as a "rule" which makes an assignment of a member of Y to each member of X. This is a useful idea, but it can be misleading. Consider the function g from $\mathbb{R}$ to $\mathbb{R}$ defined by $g(x) = (x+1)^2-2x-1$. The rule for g seems to be very different from the one for f in the previous paragraph, since we have to add one, square, and then subtract two other quantities. But notice that $g(0) = 0$, $g(2) = 4$, and $g(3) = 9$. In fact, for any number x, $g(x) = (x + 1)^2 - 2x - 1 = (x^2 + 2x + 1) - 2x - 1 = x^2$. This shows that f and g are identical functions, which we express simply by writing $f = g$. So, the function is something a bit more abstract than the "rule," rather it is the thing that is the same when considering the rules x^2 and $(x + 1)^2 - 2x - 1$.

2.3 Notation. We will often use the notation $f : X \to Y$ as a shorthand for the statement "f is a function from X to Y." When $f : X \to Y$, we call the set X the *domain* of f and Y is called the *co-domain* of f. The *range* of f is the subset of Y consisting of the members of Y which the function actually uses. More precisely $\operatorname{ran}(f) \overset{\text{def}}{=} \{\, f(x) : x \in \operatorname{dom}(f) \,\}$. When we define a function by a simple equation it is understood that its domain is taken to be the largest possible X which makes sense. So for $f(x)=x^2$, $\mathrm{dom}(f)=\mathbb{R}$, while for $f(x)=1/x$, $\mathrm{dom}(f)=\{\,x\in\mathbb{R}:x\neq 0\,\}$.

You should already be very familiar with the ideas of continuity and differentiability for functions from $\mathbb{R}$ to $\mathbb{R}$. You should also be familiar with the idea of composing two functions, the notion of a one-to-one function, and the concept of the inverse of a one-to-one function.

Now we finally bring vectors into the picture. This semester, we will be dealing with functions whose domains and/or ranges are sets of vectors instead of sets of numbers. In fact, this is what Calculus-III was really all about, but you might not have realized it.

2.4 Vector functions of a scalar variable. We first consider the simple case where the domain is still in $\mathbb{R}$, but the range is some $\mathbb{R}^n$. How can we define such a function? All that is needed are n functions from $\mathbb{R}$ to $\mathbb{R}$ which specify the coordinates of the range. For example, $f(x)=\big\langle x,x^2,x^3+1,\sin(x)\big\rangle$ defines a function $f:\mathbb{R}\to\mathbb{R}^4$. For a function like this, we often use $\vec{f}(x)$ instead of $f(x)$ to indicate that the values of $\vec{f}$ are vectors. However, since it is usually clear from the context where the range of a function will be, the use of the arrow on the function name is sometimes considered optional.

How can we interpret these functions? One way is to think of the domain as measuring time and the value of $\vec{f}$ as giving the position of an object in space (recall that a vector itself can be interpreted as an arrow or as a point). When we do this, we usually change the domain variable to t, and in fact $\vec{r}(t)$ is a common notation. Of course the dimension of the range must be either 2 or 3 if the object we are modeling lives in the same universe that we do. On the hand, if you are building a multi-jointed robot arm whose configuration is determined by 15 measurements, then the state of the arm at time t gives a function $\vec{s}:\mathbb{R}\to\mathbb{R}^{15}$. An alternate interpretation for these functions is to "throw away" the domain of the function and only consider its range as a subset of $\mathbb{R}^n$. Interpreted in this way, we really just have another way of viewing the "parametric" curves you first saw in Calculus-II.

2.5 Scalar functions of several variables. Next we consider the case of functions $f:\mathbb{R}^n\to\mathbb{R}$. Since the value of f is a scalar, we don't put an arrow over the function name. We can consider an element of the domain to be a single vector, in which case we might use the notation $f(\vec{r})$. Equivalently, we can explicitly show the dimension of the domain by listing the coordinates of a point in the domain with separate variables. We might use

$f(x,y)$, $f(x,y,z)$, and $f(x,y,z,w)$ for the cases where $n=2$, $n=3$, and $n=4$. It is important to realize that these are just different notations. For example, we could define $f:\mathbb{R}^2\to\mathbb{R}$ by $f(x,y)=\sqrt{x^2+y^2}$, but the equation $f(\vec{r})=\|\vec{r}\|$ defines exactly the same function. For $n>4$, we usually use subscripts on the variables instead of separate letters: $f(x_1,x_2,x_3,x_4,x_5)$. In fact, subscripts can be used even in lower dimensions.

In the case when $n=2$ the simplest (and most important) interpretation is to look at the graph of $z=f(x,y)$, which will be a surface in $\mathbb{R}^3$. When $n=3$, the corresponding "graph" would have to be placed in $\mathbb{R}^4$, which we can't visualize. So, instead we can just think of f as "assigning" a number to each point in $\mathbb{R}^3$. For example, if we measure the temperature at each point in space, we obtain a function $T(x,y,z)$. Another important example is to measure density (in, say, gm/cm^3) at each point. When $n>3$, we can no longer visualize f at all. But, for example, if we want to express how the height of the end of the robot arm mentioned above varies with the state of the arm, we would use a function $h:\mathbb{R}^{15}\to\mathbb{R}$.

You should already be very familiar with the concept of partial derivatives and of iterated integrals, which apply to functions of two or more variables.

2.6 Vector functions of several variables. We now consider the general case of functions $f:\mathbb{R}^n\to\mathbb{R}^m$, which are a main focus for this course. It is easy enough to define such functions by combining the ideas from 2.4 and 2.5. We just need to give m scalar functions each of which has n variables. For example, $f(x,y)=\big\langle x^2,y^3,x+y\big\rangle$ defines $f:\mathbb{R}^2\to\mathbb{R}^3$, and $f(x,y,z)=\big\langle x^2z,\,x+y-z\big\rangle$ defines $f:\mathbb{R}^3\to\mathbb{R}^2$. Note that these definitions can also be written as $x^2\vec{\imath}+y^3\vec{\jmath}+(x+y)\vec{k}$ and $x^2z\,\vec{\imath}+(x+y-z)\vec{\jmath}$ (although the second equation could be accidentally interpreted as having a 3-dimensional range).

Visualizing these functions can be a challenge. First we will consider the cases where n and m are equal and no bigger than 3, i.e., $f:\mathbb{R}^2\to\mathbb{R}^2$ and $f:\mathbb{R}^3\to\mathbb{R}^3$. One can think of these functions in a simple "pointwise" fashion—f "moves" each point of the plane (or $\mathbb{R}^3$) into a new position. The term "move" is a little misleading, since the motion isn't smooth or continuous with respect to time. Instead, we think each point $\vec{r}$ jumping into a new position $f(\vec{r})$. Actually, you have been using this interpretation since high school. To make it more obvious, lets use the letters r and θ (instead of x and y or $\vec{p}$) for the coordinates of the domain, and consider the function defined by $f(r,\theta)=\langle r\cos(\theta),r\sin(\theta)\rangle$. This is just the "coordinate transformation" which corresponds to polar coordinates. As a simple exercise, you should also write down the functions which correspond to cylindrical and spherical coordinates. Since complex numbers are really just another interpretation of two-dimensional vectors, complex-valued functions are usually also viewed in this way.

However, there is another visualization for $f : \mathbb{R}^2 \to \mathbb{R}^2$ and $f : \mathbb{R}^3 \to \mathbb{R}^3$ which is much more important for this class. We take the domain of f to represent points, but interpret the range of f as vectors visualized as *arrows.* We imagine the base of each arrow $f(\vec{r})$ as being placed at the point $\vec{r}$. The result is a picture of the plane (or $\mathbb{R}^3$) which is continuously covered up with arrows of varying length pointing in various directions. When we interpret f in this way we call f a *in vector field.* We also usually use $\vec{f}(\vec{r})$ or $\vec{f}(x, y, z)$ to emphasize that we are using this interpretation. It is important to realize that the arrows don't really overlap or intersect—we think of each one as being independent of the other. It would require 4 or 6 dimensions to really "draw" the arrows in an independent way. Since it is basically impossible to picture what these higher dimensions look like, we have to make due with just pretending that the arrows don't cross each other.

A vector field is exactly what you need in order to model the situation of a varying force at each point in space. For this reason, our book concentrates on the case of the electric field, which is a simple situation where this kind of force occurs. Another important use for vector fields is to measure varying velocity at each point in space (i.e., velocity as a vector, not just speed). This occurs when you want to model the flow of a fluid. The power (and perhaps even beauty) of mathematics is that the same mathematical object can be used to describe very different sorts of problems. So the concepts of vector calculus will apply equally well to electromagnetic theory and to fluid mechanics. For a somewhat challenging exercise, take functions for the polar, cylindrical, and spherical coordinate transformations, and interpret them as vector fields. What do these fields look like?

As a final case, consider $f : \mathbb{R}^2 \to \mathbb{R}^3$. To visualize such a function, we use the idea of 2.4 and "throw away" the domain and just concentrate on the range. Since the domain is just two-dimensional, the range is likely to be a "thin" two-dimensional surface in three-dimensional space. This sort of a description is called a *parametric surface.* For example, $f(r, s) = \langle 2r - s, s + 1, 3r + s + 1\rangle$ would describe just a flat plane (since all the coordinates change linearlly). On the other hand, see if you can show that $f(r, s) = \langle \sin(r)\cos(s), \sin(r)\sin(s), \cos(r)\rangle$ is a parametric representation for the unit sphere.

2.7 Even more general functions.

To finish this up, we briefly introduce a more abstract example. Let V be the vector space example from the end of the first handout: V denotes the set of all continuous functions from $\mathbb{R}$ to $\mathbb{R}$. Consider the equation

$$f(t)(x) \overset{\text{def}}{=} t^3x - \frac{1}{t^2+1}$$

If we had written $f(t,x)$, then we could interpret the equation on the right side as defining $f : \mathbb{R}^2 \to \mathbb{R}$. But with the above notation, we consider the equation as instead defining $f : \mathbb{R} \to V$. In other words, for each number $t \in R$, we fix t and consider the equation $t^3x - \frac{1}{t^2+1}$ as defining a function in V by using x as the variable. Note that $f(t)$ is always a function whose graph is a line, even though t^2 and t^3 appear in the definition!

Here is a more natural example. Fix the simple parabola $y = x^2$. Now, for each number t, let $f(t)$ be the *function* whose graph is the line tangent to the parabola at the point on the graph whose x-coordinate is t. As an exercise, you should write out a definition like the one above for this function f.

What has been hidden throughout Calculus I, II, and III is that derivatives are really functions of this type. However functions whose graphs are lines are determined by just two numbers—the slope and y-intercept. From many points of view, the slope of a tangent line is far more important than its intercept, so we can ignore the latter and just consider the simpler "linear" function of the form $y = mx$ whose graph is parallel to the corresponding tangent line. Since these functions are entirely determined by one number (the slope) we can get away with saying that the derivative f' is just a function from $\mathbb{R}$ to$\mathbb{R}$. We would much rather deal with $f'(x) = 2x$ than $f'(x)(t) = (2x)(t)$. But the vector situation is more complex, and really the *best* way to consider derivatives in higher dimensions is in this abstract setting. For now, though, we'll concentrate on studying vector fields. We'll come back to this topic in more detail later in the semester.