Tennessee Technological University
Mathematics Department

MATH 4850-5850: Computational Algebraic Geometry I

I. COURSE DESCRIPTION FROM CATALOG:

Affine varieties and polynomial ideals, Groebner bases, elimination theory, Hilbert's Nullstellensatz, Zariski closure, decomposition into irreducible varieties.
Lec. 3-3. Cr. 3-3.

II. PREREQUISITE(S):

MATH 4850/5850: C or better in MATH 2010, and C or better in MATH 3400 or equivalent (or consent of instructor for MATH 5850). Additional recommended prerequisite: MATH 3510 or any other 4000/5000 level mathematics course in which proofs are required.

III. COURSE OBJECTIVE(S):

The main purpose of this sequence is to introduce undergraduates and beginning graduate students to some interesting ideas and computational algorithms in algebraic geometry and commutative algebra. Until some forty years ago these topics involved a lot of abstract mathematics and were primarily taught in graduate school. Since about the 1960's thanks to the discovery by Buchberger and Hironaka of new algorithms for efficient manipulating systems of polynomial equations and the development of computers fast enough to run these algorithms, it has become possible to investigate complicated examples that would be impossible by hand. This has changed the practice of much research in algebraic geometry, which now relies on the computational approach, and it has also enhanced the importance of the subject and its methods for computer scientists and engineers.

The growing importance of these computational techniques which are virtually unknown outside of mathematics and very theoretical computer science warrants their introduction into the undergraduate and graduate mathematics curriculum. Many undergraduates, including those attending TTU, enjoy the concrete flavor that a computational yet mathematically very sound emphasis brings to this subject. Applications – many of which are to be discussed in this sequence - range from computational abstract algebra such as, for example, invariant theory of finite groups, cryptography, automatic geometric theorem proving, symmetric polynomials, geometry of quadric hypersurfaces, through theoretical computer science involving study of new algorithms, their pseudocodes and eventual programming, computer vision and projective geometry, to problems in robotics such as, for example, geometric description of robots, the forward and the inverse kinematic problems, motion planning, parallel lines, etc. At the same time, one can do some substantial mathematics including the Hilbert Basis Theorem, Elimination Theory, the Nullstellensatz, and the Bezout's Theorem.

The sequence assumes that the students have access to a computer algebra system such as Maple, FGb, Mathematica, AXIOM, Singular, REDUCE, or Magma. TTU already owns the Maple license whereas FGb, AXIOM and Singular are free. Purchase of high-powered Magma is envisioned through a QEP proposal that has been submitted by the Mathematics Department but is not necessary to the success of this new sequence.

Introduction of this sequence would also facilitate involving many mathematics, computer science, and engineering undergraduate and graduate students in the state-of-the art computational research with a large dose of theoretical and practical flavors currently not available at TTU.

Graduate students will be assigned additional readings on more advanced topics, applications, or programming exercises chosen by the instructor and fitting their major interests. They may be asked to present these topics in class or to submit them as additional graded work.

IV. TOPICS TO BE COVERED:

MATH 4850/5850:

1. Geometry, Algebra, and Algorithms
    1. Polynomials and Affine Space
    2. Affine Varieties
    3. Parametrizations of Affine Varieties
    4. Ideals
    5. Polynomials of One Variable

2. Groebner Bases
    1. Introduction
    2. Orderings on the Monomials in $k[x_1, ..., x_n]$
    3. A Division Algorithm in $k[x_1, ..., x_n]$
    4. Monomial Ideals and Dickson's Lemma
    5. The Hilbert Basis Theorem and Groebner Bases
    6. Properties of Groebner Bases
    7. Buchberger's Algorithm
    8. First Applications of Groebner Bases
    9. (Optional) Improvements on Buchberger's Algorithm

3. Elimination Theory
    1. The Elimination and Extension Theorems
    2. The Geometry of Elimination
    3. Implicitization
    4. Singular Points and Envelopes
    5. Unique Factorization and Resultants
    6. Resultants and the Extension Theorem

4. The Algebra-Geometry Dictionary
    1. Hilbert's Nullstellensatz
    2. Radical Ideals and the Ideal-Variety Correspondence
    3. Sums, Products, and Intersections of Ideals
    4. Zariski Closure and Quotients of Ideals

5. Irreducible Varieties and Prime Ideals
6. Decomposition of a Variety into Irreducibles
7. (Optional) Primary Decomposition of Ideals
8. Summary

V. ADDITIONAL INFORMATION:

While the prerequisites listed for MATH 4850/5850 are sufficient, completing one of the following mathematics courses in which abstract proofs are required is strongly recommended: MATH 3510, 3520, 4050/5050, 4110/5110, 4310/5310, 4350/5350, 4360/5360, 4530/5540, 4750/5750. However, the course is self contained and does not require knowledge of abstract algebra. It does require basic linear algebra and some familiarity with mathematical proofs typically gained, at a minimum, in a discrete mathematics type course, e.g. MATH 2610, or in a proofs course, e.g., MATH 3400, or equivalent. Some basic knowledge of structured programming will be useful.

Access to a Computer Algebra system such as Maple, Mathematica, AXIOM, REDUCE, or Singular is required in order to test algorithms and do computational exercises.

Graduate credit is earned on the basis of additional work required by the instructor per TTU Graduate Catalog. For example graduate students can be expected to present topics and applications in class, complete computational and programming projects, etc.

Topics in MATH 4850/5850 consist of Chapters 1 – 4 from [1] whereas topics in MATH 4860/5860 consist of Chapters 5, 8, 9 and either 6 or 7 depending on the audience and interests of the instructor. Chapters 1 – 4 need to be covered in succession. After Chapter 4, one can immediately cover any one of the Chapters 5, 6, 7, or 8 with Chapter 9 following Chapter 5 or 8. It is recommended to cover material in MATH 4860/5860 as follows: Chapter 5, 8, 9 followed by Chapter 6 and/or Chapter 7.

VI. POSSIBLE TEXTS AND REFERENCES:

Primary:

1. David A. Cox, John B. Little, Donal B. O'Shea, *"Ideals, Varieties, and Algorithms – An Introduction to Computational Algebraic Geometry and Commutative Algebra"*, Springer, 3/e edition (March 2, 2007) New York, 2007. ISBN-10: 0387356509

Supplementary:

2. David A. Cox, John B. Little, Donal B. O'Shea, *"Using Algebraic Geometry,"* Springer; 1 edition (August 13, 1998) ISBN: 0387207066
3. Gert-Martin Greuel, Gerhard Pfister, *"A Singular Introduction to Commutative Algebra,"* Springer; 1 edition (October 3, 2002), ISBN: 3540428976

VII. ANY TECHNOLOGY THAT MAY BE USED:

- Maple 11 with its packages for Groebner and FGb for basis computations. FGb is available **free** from http://fgbrs.lip6.fr/jcf/Software/FGb/index.html and it runs as an add-on to Maple. Maple and FGb are already installed in Bruner 305 lab.
- Singular - A Computer Algebra System for polynomial computations with special emphasis on the needs of commutative algebra, algebraic geometry, and singularity theory. Singular is available **free** from http://www.singular.uni-kl.de/ and can be interfaced with Maple.
- AXIOM – A general purpose system for doing mathematics by computer. It is especially useful for mathematical research and for development of mathematical algorithms. AXIOM is available **free** from http://wiki.axiom-developer.org/FrontPage and can be interfaced with Maple.
- The Magma Computational Algebra System for Algebra, Number Theory, and Geometry. See http://magma.maths.usyd.edu.au/, however this software requires a license of $1,725 for up to four single-processor PC's. Special discounts are available too.
- The primary text contains appendices showing how to use AXIOM, Maple, Mathematica, REDUCE, Singular.

The Mathematics Department Computer Lab in Bruner 305 and/or the parallel lab in the Department of Computer Science will be used in this course.